Reference-wave solutions for the high-frequency field in random media

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Ray trajectories, as has been shown in the recently formulated stochastic geometrical theory of diffraction, play an important role in determining the propagation properties of high-frequency wave fields and their statistical measures in complicated random environments. The field at the observer can be presented as the superposition of a variety of field species arriving at the observer along multiple ray trajectories resulting from boundaries and scattering centers embedded into the random medium. In such situations the intensity products from which the average intensity measures can be constructed and which, in general, are presented as even products of the total field, will contain sums of products of mixed field species arriving along different ray trajectories. For computations of the statistical measures of the field it is desirable, therefore, to possess a solution for the high-frequency field propagating along an isolated ray trajectory. The main concern of this work is the construction of high-frequency asymptotic propagators, relating the values of the random field and its statistical measures at some observation plane to their source (actual or virtual) distributions at the initial plane. For this reason a reference-wave method was developed to obtain an approximate solution of the parabolic wave equation in a homogeneous background random medium.

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I. INTRODUCTION

Wave propagation taking place either in the natural environments or in the artificial structures is usually accompanied by random phenomena caused by the fluctuations of the medium's parameters. While such problems arise in various areas of physics, the most appreciable theoretical and experimental achievements have been made in the case of continuously fluctuating media when the medium's fluctuations can be considered as large-scale compared to the radiation wavelength. These achievements have been stimulated by the practical importance of such topics as laser beam propagation in a turbulent atmosphere and acoustic wave propagation in a fluctuating underwater sound channel. At the present time the subject is supported by a wide theoretical background that includes sophisticated analytical and numerical methods described and summarized in a number of monographs (we mention only a few of them [1–3]). In the line-of-sight case when the radiation from the source approaches the observer along a straight line (or a single curved ray in the case of an inhomogeneous background medium) the problem can be described by studying the propagation of the statistical moments of the field. The most important are the second-order moment related to the coherence properties and average intensity of the field and the fourth-order moment related to the correlation properties of the field intensity.

An increase in the propagation ranges and the need to operate with fields having greater spatial and angular extents require that the complexity of the propagating environments and the resulting multipath effects induced by the scattering of the field by boundaries and scattering centers to be accounted for. The locations of such obstacles can be either deterministic or random. In the latter case additional statistical treatment is required. Investigations of the field structure in complicated environments becomes especially important in the modeling of modern mobile and satellite communication channels operating in the millimeter wave range. The prescriptions of how to incorporate all these effects into stochastic propagation are given in a stochastic geometrical theory of diffraction (SGTD) [4,5] which has been especially formulated in order to deal with such types of phenomena. The SGTD is based on the localization concept according to which the high-frequency fields are concentrated along the ray trajectories specified by the deterministic GTD, and, therefore, can be transported along these trajectories by taking account of the effect of random inhomogeneities on their phase and the amplitude. As in the deterministic GTD, the field at the observer can comprise a number of field species arriving along different ray trajectories resulting from the reflection, refraction, and (or) diffraction of the local plane-wave fields by boundaries, inhomogeneities, and (or) scattering centers [6,7]. As in the line-of-sight case the statistical properties of the observed intensity patterns can be derived from the analysis of the statistical intensity moments. However because of the multipath arrival, the expressions for the intensity moments, being even order products of the total field at the observer, will contain odd products of the separate ray-field species. Moreover, the radiation portions propagating along different rays can traverse the same spatial regions, which requires consideration of their correlation. Therefore, it would be useful to possess a field solution that accounts for the information accumulated by the propagating field along its propagation path. The derivation of such a solution is one of the main concerns of this work. Our solution strategy is based on the development of a reference wave method (RWM). The methodology is based on defining a paired field measure as a product of an unknown field propagating in a disturbed medium and its a complex conjugate component propagating in a medium without random fluctuations. The solution of the deterministic equation can

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be usually obtained by conventional methods. Defining paired field measures and extensions to higher dimensional spaces is stimulated by the advantages that they open in analyzing field structure. In particular, this allows us to deal simultaneously with location in a configuration space together with defining the ray slope and the spectral properties of the radiation. In addition, performing a proper scaling gives us the ability to emphasize "slow" and "fast" variables and to define expansion parameters with the subsequent application of sophisticated asymptotic techniques [8–11]. Once a solution of the equation for the paired field measure is obtained, the solution of the unknown field can be easily extracted from the paired solution in an explicit form if one knows the solution of the deterministic component. The reference-wave method has already been applied successfully to the parabolic-type equations [12].

II. THE REFERENCE-WAVE SOLUTION

The starting point of our analysis is the scalar Helmholtz equation for the time-harmonic field $U(\boldsymbol{R})$:

$$\nabla^2 U(\boldsymbol{R}) + k^2 N^2(\boldsymbol{R}) U(\boldsymbol{R}) = 0. \tag{1}$$

Here $\boldsymbol{R}$ measures the location in three-dimensional space and can be represented in different curvilinear coordinate systems, $N(\boldsymbol{R}) = 1 + n(\boldsymbol{R})$ is the refractive index of the medium, which consists of a unit background part and a weak random part $n(\boldsymbol{R})$, $(|n(\boldsymbol{R})| \ll 1)$.

As mentioned above, propagation of high-frequency time-harmonic signals in spatially inhomogeneous media takes place along the geometrical ray trajectories representing the paths of energy flux transfer. Our concern is to construct a solution for the high-frequency random field, which is supposed to contain information about the medium refractive index along the propagation path. In this work we restrict ourselves to a homogeneous background random medium and base our solution on the parabolic approximation along a straight background ray thereby extracting from the high-frequency field the main phase variation along some reference ray path [1–3]:

$$U(\boldsymbol{r}, \sigma) = u(\boldsymbol{r}, \sigma) \exp(ik\sigma). \tag{2}$$

Here the propagation of the random parabolic wave amplitude $u(\boldsymbol{r}, \sigma)$ is described in a ray-centered coordinate system $\boldsymbol{R} = \{\boldsymbol{r}, \sigma\}$, where the two-dimensional radius-vector $\boldsymbol{r}$ measures the location in a rectangular coordinate system perpendicular to the straight reference ray and the coordinate σ measures the range along that ray (for definition of such coordinate system see, for example, Refs. [4] and [5] and also one of the ray trajectories in Figs. 1 and 2). Substituting Eq. (2) into Eq. (1) and neglecting the "slow" range derivatives, we arrive at the parabolic equation for the propagator of the reduced wave amplitude:

$$\frac{\partial g_1(\boldsymbol{r}_1, \sigma | \boldsymbol{r}_{10}, \sigma_0)}{\partial \sigma} = \frac{i}{2k} \nabla^2_{\boldsymbol{r}_1} g_1(\boldsymbol{r}_1, \sigma | \boldsymbol{r}_{10}, \sigma_0) + ikn(\boldsymbol{r}_1, \sigma) g_1(\boldsymbol{r}_1, \sigma | \boldsymbol{r}_{10}, \sigma_0), \tag{3}$$

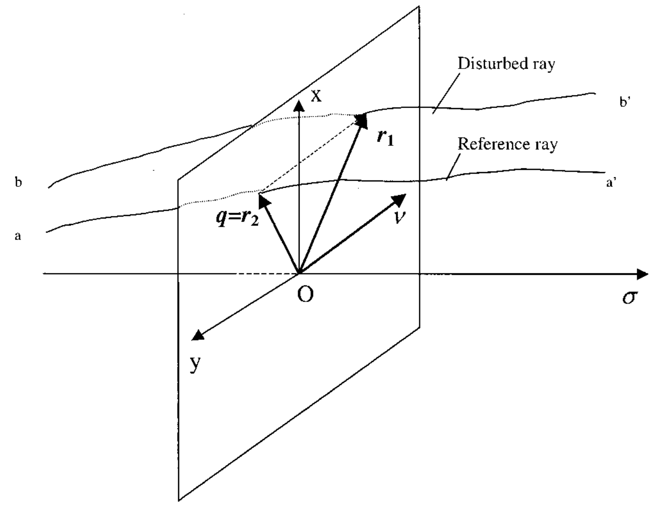

FIG. 1. Transformation to reference and displacement coordinates, Eq. (7).

which is solved with the initial condition:

$$g_1(\boldsymbol{r}_1, \sigma_0 | \boldsymbol{r}_{10}, \sigma_0) = \delta(\boldsymbol{r}_1 - \boldsymbol{r}_{10}). \tag{3a}$$

Since Eq. (3) contains a random function $n(\mathbf{r}, \sigma)$, its solutions will be also random functions. Originally the development of solutions of the SGTD propagators was closely related to the paired field measures. The fact that the propagation of these measures is described in an extended space allowed us to emphasize "slow" and "fast" variables with the possibility of applying sophisticated multiscale asymptotic techniques [8–11]. Here, we suggest the application of similar methods in order to extract the solution for the random field itself. In parallel to the propagation in a random medium, we consider also the propagation of a deterministic wave in a medium in the absence of the refractive index fluctuations. This is described by the equation:

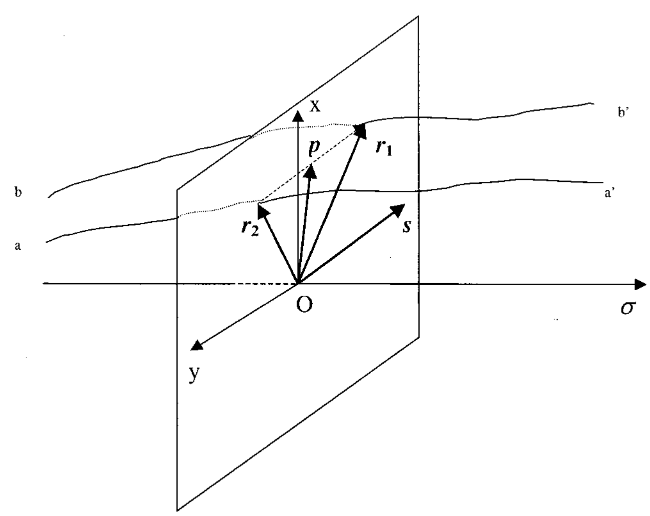

FIG. 2. Transformation to center of mass and difference coordinates, Eq. (25).

$$\frac{\partial u_2(\boldsymbol{r}_2,\sigma)}{\partial\sigma}=\frac{i}{2k}\nabla^2_{\boldsymbol{r}_2}u_2(\boldsymbol{r}_2,\sigma),\quad u_2(\boldsymbol{r}_2,\sigma_0)=u_0(\boldsymbol{r}_2). \tag{4}$$

Defining a product

$$\Phi(\boldsymbol{r}_1,\boldsymbol{r}_2,\sigma)=g_1(\boldsymbol{r}_1,\sigma|\boldsymbol{r}_{10},\sigma_0)u_2^*(\boldsymbol{r}_2,\sigma), \tag{5}$$

and using the standard procedure, we arrive at the equation for Φ:

$$\frac{\partial\Phi(\boldsymbol{r}_1,\boldsymbol{r}_2,\sigma)}{\partial\sigma}=\frac{i}{2k}(\nabla^2_{\boldsymbol{r}_1}-\nabla^2_{\boldsymbol{r}_2})\Phi(\boldsymbol{r}_1,\boldsymbol{r}_2,\sigma)+ikn(\boldsymbol{r}_1,\sigma)\Phi(\boldsymbol{r}_1,\boldsymbol{r}_2,\sigma), \tag{6}$$

$$\Phi(\boldsymbol{r}_1,\boldsymbol{r}_2,\sigma_0)=\delta(\boldsymbol{r}_1-\boldsymbol{r}_{10})u_{20}(\boldsymbol{r}_2). \tag{6a}$$

Defining the product $\Phi(\boldsymbol{r}_1,\boldsymbol{r}_2,\sigma)$ in Eq. (5) allows to emphasize the phase differences between the desired solution for $g_1(\boldsymbol{r}_1,\sigma_0|\boldsymbol{r}_{10},\sigma)$ and the reference wave $u_2(\boldsymbol{r}_2,\sigma)$. As will be shown later, the propagation of the reference wave is not necessarily carried along the deterministic ray trajectories. It can be carried also along the random rays determined by the random medium. In order to emphasize "slow" and "fast" variables in Eq. (6), we introduce new transverse coordinates (see Fig. 1)

$$\boldsymbol{q}=\boldsymbol{r}_2,\quad \boldsymbol{q}_0=\boldsymbol{r}_{20}, \tag{7a}$$

$$\boldsymbol{\nu}=\boldsymbol{r}_1-\boldsymbol{r}_2,\quad \boldsymbol{\nu}_0=\boldsymbol{r}_{10}-\boldsymbol{r}_{20}, \tag{7b}$$

where $\boldsymbol{q}$ is the coordinate of an undisturbed reference ray, while the difference vector coordinate $\boldsymbol{\nu}$ describes the disturbed ray displacement with respect to the reference ray trajectory. We note that the transformations in Eq. (7) are nonsymmetric with the respect to the original coordinates $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. This asymmetry allows to preserve the phase information caused by the changes in the refractive index along the propagation path. Applying these transformations leads to the following equation for the function $\Pi(\boldsymbol{\nu},\boldsymbol{q},\sigma|\boldsymbol{\nu}_0,\boldsymbol{q}_0,\sigma_0)=\Phi(\boldsymbol{\nu}+\boldsymbol{q},\boldsymbol{q},\sigma|\boldsymbol{\nu}_0+\boldsymbol{q}_0,\boldsymbol{q}_0,\sigma_0)$:

$$\frac{\partial\Pi(\boldsymbol{\nu},\boldsymbol{q},\sigma|\boldsymbol{\nu}_0,\boldsymbol{q}_0,\sigma_0)}{\partial\sigma}=\frac{i}{k}\boldsymbol{\nabla}_{\boldsymbol{\nu}}\cdot\boldsymbol{\nabla}_{\boldsymbol{q}}\Pi(\boldsymbol{\nu},\boldsymbol{q},\sigma|\boldsymbol{\nu}_0,\boldsymbol{q}_0,\sigma_0)-\frac{i}{2k}\nabla^2_{\boldsymbol{q}}\Pi(\boldsymbol{\nu},\boldsymbol{q},\sigma|\boldsymbol{\nu}_0,\boldsymbol{q}_0,\sigma_0)+ikn(\boldsymbol{\nu}+\boldsymbol{q},\sigma)\times\Pi(\boldsymbol{\nu},\boldsymbol{q},\sigma|\boldsymbol{\nu}_0,\boldsymbol{q}_0,\sigma_0), \tag{8}$$

with the source condition

$$\Pi(\boldsymbol{\nu},\boldsymbol{q},\sigma_0|\boldsymbol{\nu}_0,\boldsymbol{q}_0,\sigma_0)=\delta(\boldsymbol{\nu}+\boldsymbol{q}-\boldsymbol{\nu}_0-\boldsymbol{q}_0)u_0(\boldsymbol{q}). \tag{8a}$$

To justify our approximations, we emphasize the explicit dependence of the refractive index $n(\boldsymbol{r},\sigma)=\tilde{n}(\boldsymbol{r}/\ell,\sigma/\ell)$, where ℓ is a characteristic spatial scale of the medium's fluctuations (it can be associated with the correlation length). Such scaling allows us to introduce a small expansion parameter $\varepsilon=1/(k\ell)$, which is of the order of a single scattering angle. To obtain an approximate solution, generally, we can employ the multiscale expansion procedure as developed on the previous works and applied to similar equations with distinct "fast" and "slow" variables. Instead of performing scaling of the variables, we will perform here a formal expansion into powers of inverse wave number k^{-1}.

The common way of solving Eq. (8) is to transform the function $\Pi(\boldsymbol{\nu},\boldsymbol{q},\sigma|\boldsymbol{\nu}_0,\boldsymbol{q}_0,\sigma_0)$ from the domain described by the $\boldsymbol{\nu}$-$\boldsymbol{q}$ coordinates to the phase-space $\boldsymbol{\nu}$-$\boldsymbol{\rho}$. The function $\overline{\Pi}(\boldsymbol{\nu},\rho,\sigma|\boldsymbol{\nu}_0,\rho_0,\sigma_0)$ is defined as a spectral transform:

$$\overline{\Pi}(\boldsymbol{\nu},\boldsymbol{\rho},\sigma|\boldsymbol{\nu}_0,\boldsymbol{\rho}_0,\sigma_0)=\left(\frac{k}{2\pi}\right)^2\int\int_{-\infty}^{\infty}d\boldsymbol{q}\,d\boldsymbol{q}_0\times\Pi(\boldsymbol{\nu},\boldsymbol{q},\sigma|\boldsymbol{\nu}_0,\boldsymbol{q}_0,\sigma_0)\times\exp\{-ik[\boldsymbol{\rho}\cdot\boldsymbol{q}-\boldsymbol{\rho}_0\cdot\boldsymbol{q}_0]\}. \tag{9}$$

Applying this transform to Eq. (8), we obtain equation for $\overline{\Pi}(\boldsymbol{\nu},\boldsymbol{\rho},\sigma|\boldsymbol{\nu}_0,\boldsymbol{\rho}_0,\sigma_0)$:

$$\frac{\partial\overline{\Pi}(\boldsymbol{\nu},\boldsymbol{\rho},\sigma|\boldsymbol{\nu}_0,\boldsymbol{\rho}_0,\sigma_0)}{\partial\sigma}+\boldsymbol{\rho}\cdot\boldsymbol{\nabla}_{\nu}\overline{\Pi}(\boldsymbol{\nu},\boldsymbol{\rho},\sigma|\boldsymbol{\nu}_0,\boldsymbol{\rho}_0,\sigma_0)-ikn\left(\boldsymbol{\nu}+\frac{i}{k}\boldsymbol{\nabla}_{\boldsymbol{\rho}},\sigma\right)\overline{\Pi}(\boldsymbol{\nu},\boldsymbol{\rho},\sigma|\boldsymbol{\nu}_0,\boldsymbol{\rho}_0,\sigma_0)=\frac{ik\rho^2}{2}\overline{\Pi}(\boldsymbol{\nu},\boldsymbol{\rho},\sigma|\boldsymbol{\nu}_0,\boldsymbol{\rho}_0,\sigma_0). \tag{10}$$

Expanding the function $n[\boldsymbol{\nu}+(i/k)\boldsymbol{\nabla}_{\boldsymbol{\rho}},\sigma]$ into powers of k^{-1} we obtain for the main order the following equation:

$$\frac{\partial\overline{\Pi}(\boldsymbol{\nu},\boldsymbol{\rho},\sigma|\boldsymbol{\nu}_0,\boldsymbol{\rho}_0,\sigma_0)}{\partial\sigma}+\boldsymbol{\rho}\cdot\boldsymbol{\nabla}_{\nu}\overline{\Pi}(\boldsymbol{\nu},\boldsymbol{\rho},\sigma|\boldsymbol{\nu}_0,\boldsymbol{\rho}_0,\sigma_0)+\boldsymbol{\nabla}_{\boldsymbol{\nu}}n(\boldsymbol{\nu},\sigma)\cdot\boldsymbol{\nabla}_{\boldsymbol{\rho}}\overline{\Pi}(\boldsymbol{\nu},\boldsymbol{\rho},\sigma|\boldsymbol{\nu}_0,\boldsymbol{\rho}_0,\sigma_0)=\frac{ik\rho^2}{2}\overline{\Pi}(\boldsymbol{\nu},\boldsymbol{\rho},\sigma|\boldsymbol{\nu}_0,\boldsymbol{\rho}_0,\sigma_0)+ikn(\boldsymbol{\nu},\sigma)\overline{\Pi}(\boldsymbol{\nu},\boldsymbol{\rho},\sigma|\boldsymbol{\nu}_0,\boldsymbol{\rho}_0,\sigma_0). \tag{11}$$

We note that Eq. (11) is a nonhomogeneous partial differential equation. In the homogeneous case, i.e., when the right hand side is equal to zero, this equation is similar to the transport equation for the Wigner function in the geometric approximation.

The easiest way of solving Eq. (11) is by choosing the reference source as a plane wave. Such a choice leads to the following source condition for (11):

$$\overline{\Pi}(\boldsymbol{\nu},\boldsymbol{\rho},\sigma_0|\boldsymbol{\nu}_0,\boldsymbol{\rho}_0,\sigma_0)=(2\pi/k)^2\delta(\boldsymbol{\rho}-\boldsymbol{\rho}_0)\times\exp\{ik\boldsymbol{\rho}_0\cdot(\boldsymbol{\nu}-\boldsymbol{\nu}_0)\}. \tag{12}$$

Equation (11) is a first-order partial differential equation and can be solved by the method of characteristics. The characteristic equations are given by the following system:

$$\frac{d\boldsymbol{\nu}}{d\sigma}=\boldsymbol{\rho}, \quad \boldsymbol{\nu}(\sigma)=\boldsymbol{\nu}, \tag{13}$$

$$\frac{d\boldsymbol{\rho}}{d\sigma}=\boldsymbol{\nabla}_{\boldsymbol{\nu}}n(\boldsymbol{\nu},\sigma), \quad \boldsymbol{\rho}(\sigma)=\boldsymbol{\rho}, \tag{14}$$

$$\frac{d\bar{\Pi}}{d\sigma}=\frac{ik\rho^2}{2}\bar{\Pi}+ikn(\boldsymbol{\nu},\sigma)\bar{\Pi}. \tag{15}$$

Solving Eqs. (13) and (14) we obtain the characteristics $\boldsymbol{\nu}_f(\zeta)$ and $\boldsymbol{\rho}_f(\zeta)$ as functions of the range coordinate ζ, where the boundary values $\boldsymbol{\nu}$ and $\boldsymbol{\rho}$ are specified at the observation plane σ. Actually, when the boundary conditions for $\boldsymbol{\nu}$ and $\boldsymbol{\rho}$ are determined at the observation plane we can express $\boldsymbol{\nu}_f(\zeta)$ by

$$\boldsymbol{\nu}_f(\zeta)=\boldsymbol{\nu}-\int_{\zeta}^{\sigma}d\xi\rho_f(\xi). \tag{16}$$

Using these solutions in Eq. (15) leads to

$$\begin{aligned}&\bar{\Pi}(\boldsymbol{\nu},\boldsymbol{\rho},\sigma|\boldsymbol{\nu}_0,\boldsymbol{\rho}_0,\sigma_0)\\&\quad=\bar{\Pi}(\boldsymbol{\nu}_f(\sigma_0),\boldsymbol{\rho}_f(\sigma_0),\sigma_0|\boldsymbol{\nu}_0,\boldsymbol{\rho}_0,\sigma_0)\\&\qquad\times\exp\left(\frac{ik}{2}\int_{\sigma_0}^{\sigma}d\zeta\,\rho_f^2(\zeta)\right)\\&\qquad\times\exp\left\{ik\int_{\sigma_0}^{\sigma}d\zeta\,\tilde{n}\left(\boldsymbol{\nu}-\int_{\zeta}^{\sigma}d\xi\,\boldsymbol{\rho}_f(\xi),\zeta\right)\right\}.\end{aligned} \tag{17}$$

Applying the inverse transform to Eq. (17) with the initial condition (8a), and performing the integration with respect to the $\boldsymbol{\rho}_0$ variable, we obtain

$$\begin{aligned}&\Pi(\boldsymbol{\nu},\boldsymbol{q},\sigma|\boldsymbol{\nu}_0,\boldsymbol{q}_0,\sigma_0)\\&\quad=\left(\frac{k}{2\pi}\right)^2\int\int_{-\infty}^{\infty}d\boldsymbol{\rho}\exp\left\{ik\boldsymbol{\rho}_f(\sigma_0)\cdot\left[\boldsymbol{\nu}-\int_{\sigma_0}^{\sigma}d\xi\,\boldsymbol{\rho}_f(\xi)-\boldsymbol{\nu}_0\right]\right\}\\&\qquad\times\exp\{ik[\boldsymbol{\rho}\cdot\boldsymbol{q}-\boldsymbol{\rho}_f(\sigma_0)\cdot\boldsymbol{q}_0]\}\exp\left(\frac{ik}{2}\int_{\sigma_0}^{\sigma}d\zeta\,\rho_f^2(\zeta)\right)\\&\qquad\times\exp\left\{ik\int_{\sigma_0}^{\sigma}d\zeta\,\tilde{n}\left(\boldsymbol{\nu}-\int_{\zeta}^{\sigma}d\xi\,\boldsymbol{\rho}_f(\xi)\zeta\right)\right\}.\end{aligned} \tag{18}$$

Finally, we set $\boldsymbol{q}=\boldsymbol{0}$, $\boldsymbol{q}_0=\boldsymbol{0}$, $\boldsymbol{\nu}=\boldsymbol{r}$, and $\boldsymbol{\nu}_0=\boldsymbol{r}_0$. According to the definitions (7), the reference plane wave propagates along a straight line connecting $\boldsymbol{q}=\boldsymbol{0}$ and $\boldsymbol{q}_0=\boldsymbol{0}$. Then, extracting the unit value plane-wave solution, we arrive at the expression for the field propagator:

$$\begin{aligned}g(\boldsymbol{r},\sigma|\boldsymbol{r}_0,\sigma_0)&=\left(\frac{k}{2\pi}\right)^2\int\int_{-\infty}^{\infty}d\boldsymbol{\rho}\exp\left\{ik\boldsymbol{\rho}_f(\sigma_0)\cdot\left[\boldsymbol{r}-\int_{\sigma_0}^{\sigma}d\xi\,\boldsymbol{\rho}_f(\xi)-\boldsymbol{r}_0\right]\right\}\exp\left(\frac{ik}{2}\int_{\sigma_0}^{\sigma}d\zeta\,\boldsymbol{\rho}_f^2(\zeta)\right)\\&\quad\times\exp\left\{ik\int_{\sigma_0}^{\sigma}d\xi\,\tilde{n}(\boldsymbol{r}_f(\zeta),\zeta)\right\}\end{aligned} \tag{19}$$

with

$$\boldsymbol{r}_f(\zeta)=\boldsymbol{r}-\int_{\tau}^{\sigma}d\xi\,\boldsymbol{\rho}_f(\xi). \tag{20}$$

Equation (20) represents the desired reference-wave solution. For its application in practical cases one has to develop a procedure for the averaging of the quantities in the integrand. Analyzing Eqs. (19) and (20) we note that, in general, the expression for the field propagator is not symmetric with respect to the $\boldsymbol{r}$ and $\boldsymbol{r}_0$ coordinates. The solutions of the characteristic equations require the boundary condition for $\boldsymbol{\rho}$ to be set at the range plane σ. Since, in principle, both coordinates $\boldsymbol{r}$ and $\boldsymbol{r}_0$ have to appear in Eq. (19) symmetrically, we can write an equivalent expression for the field propagator when the characteristic equations are solved subject to the boundary conditions at the σ_0 plane, replacing Eq. (20) by

$$\boldsymbol{r}_f(\zeta)=\boldsymbol{r}_0+\int_{\sigma_0}^{\zeta}d\xi\,\boldsymbol{\rho}_f(\xi), \tag{21}$$

and the $\boldsymbol{\rho}$ integration to the integration over $\boldsymbol{\rho}_0$. The expression for the field propagator can be simplified in some cases. Specifying the boundary values for $\boldsymbol{r}_f(\zeta)$ and $\boldsymbol{\rho}_f(\zeta)$ at the observer and solving the characteristic equations for the average values, we obtain straight ray trajectories:

$$\boldsymbol{r}_f(\zeta)=\boldsymbol{r}+\boldsymbol{\rho}(\sigma-\zeta), \quad \boldsymbol{\rho}_f=\boldsymbol{\rho}, \tag{22}$$

which can be used in Eq. (19), leading to the approximate solution for the field propagator:

$$\begin{aligned}g(\boldsymbol{r},\sigma|\boldsymbol{r}_0,\sigma_0)&=\left(\frac{k}{2\pi}\right)^2\int\int_{-\infty}^{\infty}d\boldsymbol{\rho}\exp[ik\boldsymbol{\rho}\cdot(\boldsymbol{r}-\boldsymbol{r}_0)]\\&\quad\times\exp\left\{-\frac{ik\rho^2(\sigma-\sigma_0)}{2}+ik\int_{\sigma_0}^{\sigma}d\zeta\,n(\boldsymbol{r}+\boldsymbol{\rho}(\sigma-\zeta),\zeta)\right\}.\end{aligned} \tag{23}$$

The expression in Eq. (23) is equal to the phase approximation of the Huygens-Kirchoff Method [13]. This solution has been obtained phenomenologically, and, as is well known, has limited applicability in the analysis of higher-order correlation measures. In order to make more suitable approximations, we propose that some of the deficiencies of these phenomenological solutions arise because of the violation of the uncertainty principle, when in the straight ray trajectories the slope and the position of the ray are stated simultaneously in the same range plane.

In order to include such an uncertainty in the description of the high-frequency propagation process there is a need for the choice of a proper pair of coordinates related to the spatial location and to the slope of geometrical ray trajectories. The above quantities can be introduced analytically into the field measures only if one considers a higher dimensional space. For that reason, we define a paired field measure called a two-point function (TPF)

$$\Gamma(\boldsymbol{p},\boldsymbol{s},\sigma)=u\left(\boldsymbol{p}+\frac{\boldsymbol{s}}{2},\sigma\right)u^{*}\left(\boldsymbol{p}-\frac{\boldsymbol{s}}{2},\sigma\right), \tag{24}$$

where

$$\boldsymbol{p}=\frac{\boldsymbol{r}_1+\boldsymbol{r}_2}{2},\quad \boldsymbol{s}=\boldsymbol{r}_1-\boldsymbol{r}_2 \tag{25}$$

are transverse sum and difference coordinates (see Fig. 2). The TPF propagator is a product of field propagators derived in Eq. (20):

$$g_2(\boldsymbol{p},\boldsymbol{s},\sigma|\boldsymbol{p}_0,\boldsymbol{s}_0,\sigma_0)=g\left(\boldsymbol{p}+\frac{\boldsymbol{s}}{2},\sigma\middle|\boldsymbol{p}_0+\frac{\boldsymbol{s}_0}{2},\sigma_0\right)g^{*}\left(\boldsymbol{p}-\frac{\boldsymbol{s}}{2},\sigma\middle|\boldsymbol{p}_0-\frac{\boldsymbol{s}_0}{2},\sigma_0\right). \tag{26}$$

We substitute the expression (20) for g into Eq. (26), and introduce two new coordinate functions, which are linear combinations of $\boldsymbol{\nu}_{f1}$ and $\boldsymbol{\nu}_{f2}$

$$\boldsymbol{p}_f(\zeta)=\frac{\boldsymbol{r}_{f1}(\zeta)+\boldsymbol{r}_{f2}(\zeta)}{2},\quad \boldsymbol{s}_f(\zeta)=\boldsymbol{r}_{f1}(\zeta)-\boldsymbol{r}_{f2}(\zeta), \tag{27}$$

and linear combinations $\boldsymbol{\rho}_{f1}$ and $\boldsymbol{\rho}_{f2}$ of the spectral variables appearing in each product term:

$$\boldsymbol{\rho}_f(\zeta)=\frac{\boldsymbol{\rho}_{f1}(\zeta)+\boldsymbol{\rho}_{f2}(\zeta)}{2},\quad \boldsymbol{\eta}_f(\zeta)=\boldsymbol{\rho}_{f1}(\zeta)-\boldsymbol{\rho}_{f2}(\zeta). \tag{28}$$

Performing the change of variables according to

$$\boldsymbol{\rho}=\frac{\boldsymbol{\rho}_1+\boldsymbol{\rho}_2}{2},\quad \boldsymbol{\eta}=\boldsymbol{\rho}_1-\boldsymbol{\rho}_2, \tag{29}$$

we obtain explicitly

$$\begin{aligned}g(\boldsymbol{p},\boldsymbol{s},\sigma|\boldsymbol{p}_0,\boldsymbol{s}_0,\sigma_0)=&\left(\frac{k}{2\pi}\right)^4\int\cdots\int_{-\infty}^{\infty}d\boldsymbol{\rho}\,d\boldsymbol{\eta}\\&\times\exp\{ik\boldsymbol{\eta}_f(\cdot)\cdot\lfloor\boldsymbol{p}_f(\cdot)-\boldsymbol{p}_0\rfloor\}\\&\times\exp\left\{ik\int_{\sigma_0}^{\sigma}d\zeta\,\boldsymbol{\eta}_f(\zeta)\cdot\boldsymbol{\rho}_f(\zeta)\right\}\\&\times\exp\{ik\boldsymbol{\rho}_f(\cdot)\cdot[\boldsymbol{s}_f(\cdot)-\boldsymbol{s}_0]\}\\&\times\exp\left\{ik\int_{\sigma_0}^{\sigma}d\zeta\left[n\left(\boldsymbol{p}_f(\zeta)+\frac{\boldsymbol{s}_f(\zeta)}{2},\zeta\right)-n\left(\boldsymbol{p}_f(\zeta)-\frac{\boldsymbol{s}_f(\zeta)}{2},\zeta\right)\right]\right\}.\end{aligned} \tag{30}$$

$\boldsymbol{p}_f(\zeta)$ and $\boldsymbol{s}_f(\zeta)$ are solutions of the following characteristic equations that can be easily obtained from Eqs. (13) and (14) (see the Appendix):

$$\frac{d\boldsymbol{p}}{d\sigma}=\boldsymbol{\rho}, \tag{31a}$$

$$\frac{d\boldsymbol{\rho}}{d\sigma}=\boldsymbol{\nabla}_{\boldsymbol{\rho}}n(\boldsymbol{p},\sigma), \tag{31b}$$

$$\frac{d\boldsymbol{s}}{d\sigma}=\boldsymbol{\eta}, \tag{31c}$$

$$\frac{d\boldsymbol{\eta}}{d\sigma}=\tfrac{1}{2}\{\boldsymbol{\nabla}_{\boldsymbol{p}}\cdot[\boldsymbol{\nabla}_{\boldsymbol{p}}n(\boldsymbol{p},\sigma)\cdot\boldsymbol{s}]+[\nabla_{p}^{2}n(\boldsymbol{p},\sigma)\boldsymbol{s}]\}. \tag{31d}$$

Taking into account the symmetry relations, we note that the functions $\boldsymbol{p}_f(\zeta)$ and $\boldsymbol{s}_f(\zeta)$ can be the solutions of the characteristic equations solved subject to the boundary conditions either at the range plane σ or σ_0. It was shown previously that the uncertainty relations play an important role in deriving the approximate solutions of TPF [14]. To account for the ray uncertainty, we define the Wigner distribution function

$$W(\boldsymbol{p},\boldsymbol{\rho},\sigma)=\left(\frac{k}{2\pi}\right)^2\int\int_{-\infty}^{\infty}d\boldsymbol{s}\Gamma(\boldsymbol{p},\boldsymbol{s},\sigma)\exp(ik\boldsymbol{\rho}\cdot\boldsymbol{s}), \tag{32}$$

and ambiguity function

$$A(\boldsymbol{\eta},\boldsymbol{s},\sigma)=\int\int_{-\infty}^{\infty}d\boldsymbol{p}\Gamma(\boldsymbol{p},\boldsymbol{s},\sigma)\exp(-ik\boldsymbol{\eta}\cdot\boldsymbol{p}). \tag{33}$$

If the source condition is defined by the Wigner distribution, the TPF at the observer is obtained by the following propagation relation:

$$\begin{aligned}\Gamma(\boldsymbol{p},\boldsymbol{s},\sigma)=&\int\int_{-\infty}^{\infty}d\boldsymbol{p}_0d\boldsymbol{\rho}_0W(\boldsymbol{p}_0,\boldsymbol{\rho}_0,\sigma_0)\\&\times g_W(\boldsymbol{p},\boldsymbol{s},\sigma|\boldsymbol{p}_0,\boldsymbol{\rho}_0,\sigma_0),\end{aligned} \tag{34}$$

where the propagator g_W is defined by

$$\begin{aligned}g_W(\boldsymbol{p},\boldsymbol{s},\sigma|\boldsymbol{p}_0,\boldsymbol{\rho}_0,\sigma_0)=&\left(\frac{k}{2\pi}\right)^2\int\int_{-\infty}^{\infty}d\boldsymbol{s}_0\\&\times g_2(\boldsymbol{p},\boldsymbol{s},\sigma|\boldsymbol{p}_0,\boldsymbol{s}_0,\sigma_0)\exp(ik\boldsymbol{\rho}_0\cdot\boldsymbol{s}_0)\end{aligned} \tag{35}$$

and in an explicit form can be presented as

$$g(\boldsymbol{p},\boldsymbol{s},\sigma|\boldsymbol{p}_0,\boldsymbol{\rho}_0,\sigma_0)$$
$$=\left(\frac{k}{2\pi}\right)^4\int\cdots\int_{-\infty}^{\infty}d\boldsymbol{\rho}\,d\boldsymbol{\eta}\,\delta(\boldsymbol{\rho}_0-\boldsymbol{\rho}_f(\sigma_0))$$
$$\times\exp\{ik\boldsymbol{\rho}_f(\cdot)\cdot\boldsymbol{s}_f(\cdot)\}$$
$$\times\exp\left\{ik\,\boldsymbol{\eta}_f(\cdot)\cdot\left[\boldsymbol{p}-\int_{\sigma_0}^{\sigma}d\zeta\,\boldsymbol{\rho}_f(\zeta)-\boldsymbol{p}_0\right]\right\}$$
$$\times\exp\left\{ik\int_{\sigma_0}^{\sigma}d\zeta\,\boldsymbol{\eta}_f(\zeta)\cdot\boldsymbol{\rho}_f(\zeta)\right\}$$
$$\times\exp\left\{ik\int_{\sigma_0}^{\sigma}d\zeta\left[n\left(\boldsymbol{p}_f(\zeta)+\frac{\boldsymbol{s}_f(\zeta)}{2},\zeta\right)-n\left(\boldsymbol{p}_f(\zeta)-\frac{\boldsymbol{s}_f(\zeta)}{2},\zeta\right)\right]\right\}. \quad (36)$$

The presence of the δ function in the integrand of Eq. (36) states a requirement that the boundary condition for the slope $\boldsymbol{\rho}$ has to be set at the initial plane σ_0. Therefore, at the same plane we have to set the boundary condition for the function $\boldsymbol{p}_f(\zeta)$, and the solution for the $\boldsymbol{p}_f(\zeta)$ trajectory subject to these boundary conditions is expressed as [15]

$$\boldsymbol{p}_f(\zeta)=\boldsymbol{p}_0+\int_{\sigma_0}^{\zeta}\boldsymbol{\rho}(t)\,dt,\quad \boldsymbol{\rho}(\sigma_0)=\boldsymbol{\rho}_0. \quad (37)$$

In the case when the Wigner distribution at the source plane creates the ambiguity function at the observation plane, the propagation relation is given by

$$A(\boldsymbol{\eta},\boldsymbol{s},\sigma)=\int\int_{-\infty}^{\infty}d\boldsymbol{p}_0d\boldsymbol{\rho}_0W(\boldsymbol{p}_0,\boldsymbol{\rho}_0,\sigma_0)\times g_{WA}(\boldsymbol{\eta},\boldsymbol{s},\sigma|\boldsymbol{p}_0,\boldsymbol{\rho}_0,\sigma_0), \quad (38)$$

with the propagator obtained by applying to Eq. (36) the spectral transform over the $\boldsymbol{p}$ variable:

$$g_{WA}(\boldsymbol{\eta},\boldsymbol{s},\sigma|\boldsymbol{p}_0,\boldsymbol{\rho}_0,\sigma_0)$$
$$=\left(\frac{k}{2\pi}\right)^4\int\cdots\int_{-\infty}^{\infty}d\boldsymbol{\rho}\,d\boldsymbol{\eta}\,\delta(\rho_0-\boldsymbol{\rho}_f(\sigma_0))$$
$$\times\delta(\boldsymbol{\eta}_f(\sigma)-\eta)\exp\{ik\boldsymbol{\rho}_f(\cdot)\cdot\boldsymbol{s}_f(\cdot)\}$$
$$\times\exp\left\{ik\,\boldsymbol{\eta}_f(\cdot)\cdot\left[\boldsymbol{p}-\int_{\sigma_0}^{\sigma}d\zeta\boldsymbol{\rho}_f(\zeta)-\boldsymbol{p}_0\right]\right\}$$
$$\times\exp\left\{ik\int_{\sigma_0}^{\sigma}d\zeta\,\boldsymbol{\eta}_f(\zeta)\cdot\boldsymbol{\rho}_f(\zeta)\right\}$$
$$\times\exp\left\{ik\int_{\sigma_0}^{\sigma}d\zeta\left[n\left(\boldsymbol{\nu}_f(\zeta)+\frac{\boldsymbol{s}_f(\zeta)}{2},\zeta\right)-n\left(\boldsymbol{\nu}_f(\zeta)-\frac{\boldsymbol{s}_f(\zeta)}{2},\zeta\right)\right]\right\}. \quad (39)$$

We note that the propagator g_{WA} relates two different configuration-spectral spaces. For the further approximation we replace the ray trajectories in Eq. (31) by their average values. The resulting solutions are straight trajectories. Using them in Eq. (38) and performing inverse transforms we arrive at the result derived by the multiscale expansion procedure [8,9]:

$$g(\boldsymbol{p},\boldsymbol{s},\sigma|\boldsymbol{p}_0,\boldsymbol{s}s_0,\sigma_0)$$
$$=\left(\frac{k}{2\pi}\right)^2\int\cdots\int_{-\infty}^{\infty}d\boldsymbol{\rho}_0d\boldsymbol{\eta}\exp\{ik\boldsymbol{\eta}\cdot[\boldsymbol{p}-\boldsymbol{p}_0-\boldsymbol{\rho}_0\times(\sigma-\sigma_0)]\}\exp\{ik\boldsymbol{\rho}_0\cdot(\boldsymbol{s}-\boldsymbol{s}_0)\}$$
$$\times\exp\left\{ik\int_{\sigma_0}^{\sigma}d\zeta\left[n\left(\boldsymbol{p}+\boldsymbol{\rho}_0(\zeta-\sigma_0)+\frac{\boldsymbol{s}}{2}+\frac{\mathbf{n}(\zeta-\sigma)}{2},\zeta\right)-n\left(\boldsymbol{p}+\boldsymbol{\rho}_0(\zeta-\sigma_0)-\frac{\boldsymbol{s}}{2}-\frac{\mathbf{n}(\zeta-\sigma)}{2},\zeta\right)\right]\right\}. \quad (40)$$

The ray uncertainty in the propagators (38) and (39) is accounted for by considering different phase-space configurations in the source and observation planes. We emphasize, however, that even if the trajectories in Eq. (39) are replaced by the straight rays, these are not the same rays. The slope of the rays from the source is different to the ray slopes approaching the observer. For example, the radiation emanating from the source coordinate $\boldsymbol{p}_0$ along the rays centered around the slope $\boldsymbol{\rho}_0$ creates at the observation plane the distribution characterized by transverse separation $\boldsymbol{s}$ created by independent rays having slope differences $\boldsymbol{\eta}$. Physically, this approximation accounts for the scattering of the propagating rays.

III. SUMMARY AND DISCUSSION

In this paper, we have formulated a reference-wave method applied to solve the parabolic-type wave equation. Using this method we presented solutions for the propagator governing the transport of the high-frequency field along a properly chosen deterministic ray trajectory in a randomly perturbed medium. We restricted ourselves to a medium having a homogeneous background profile, but the extension to an arbitrary background case seems to be straightforward. The desired field solution in the general case is presented as a spectral integral over various spectral contributions propagating along random ray trajectories. Approximating these trajectories by average trajectories leads to the well-known phase approximation of the Huygens-Kirchoff method. Further, we used our solution in the construction of the paired field measures associated with the coherence functions of the field. We have shown that uncertainty considerations play an important role in the construction of the statistical propagation characteristics. In order to account for the uncertainty in the high-frequency propagation there is a need to choose a proper pair of coordinates related to the spatial location and to the slope of geometrical ray trajectories. The above quantities can be introduced into the propagation process analytically only by considering a higher dimensional space, which allows us to transfer it to mixed configuration-phase-space quantities. As a starting point we defined a paired field measure called the two-point random function and its spectral transforms known as Wigner and ambiguity functions. We have shown that the ray uncertainty can be retained even while replacing the multiple trajectories by straight ray trajectories if one considers propagation between different configuration-spectral spaces. In this case our result leads to solutions that correctly represent intensity correlation characteristics. The result, as is shown, is equal to that obtained by the two-scale expansion method. These propagators were extensively applied for construction of various statistical moments and calculations of the statistical characteristics of high-frequency fields propagating in random media [9]. The fact that these propagators preserve the random information accumulated along the propagation paths, makes them suitable also for the analysis of the intensity enhancement and localization effects [8,9,14].

The further extension of the solutions presented in this work would be the development of an averaging procedure and construction of the statistical measures directly from the solutions of the propagating field itself, and not from the paired products. This will allow us to solve a number of problems not accessible before, among them analysis of the field localization effects, obtaining expressions for the multifrequency coherence functions and performing the analysis of pulsed signal propagation. First results in this direction have already been obtained [12].

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APPENDIX: RAY-TRACING EQUATIONS

Here we will derive the ray-tracing equations for the variable $\boldsymbol{p}$ and $\boldsymbol{s}$ defined by Eqs. (31) of the main text. Let us rewrite the ray-tracing Eqs. (13) and (14) for the coordinates $\boldsymbol{\nu}_i$ and the slopes $\boldsymbol{\rho}_i$, $i=1$, 2:

$$\frac{d\boldsymbol{\nu}_i}{d\sigma}=\boldsymbol{\rho}_i, \tag{A1}$$

$$\frac{d\boldsymbol{\rho}_i}{d\sigma}=\boldsymbol{\nabla}_{\boldsymbol{\nu}_i} n(\boldsymbol{\nu}_i,\sigma). \tag{A2}$$

Next we define new sum and difference coordinates

$$\boldsymbol{p}=\frac{\boldsymbol{\nu}_1+\boldsymbol{\nu}_2}{2}, \quad \boldsymbol{s}=\boldsymbol{\nu}_1-\boldsymbol{\nu}_2, \tag{A3}$$

and the slopes

$$\boldsymbol{\rho}=\frac{\boldsymbol{\rho}_1+\boldsymbol{\rho}_2}{2}, \quad \boldsymbol{\eta}=\boldsymbol{\rho}_1-\boldsymbol{\rho}_2. \tag{A4}$$

Taking the sum and difference of (A1) and (A2) according to (A3), we obtain equations for $\boldsymbol{p}$ and $\boldsymbol{s}$:

$$\frac{d\boldsymbol{p}}{d\sigma}=\boldsymbol{\rho}, \tag{A5}$$

$$\frac{d\boldsymbol{\rho}}{d\sigma}=\frac{1}{4}\frac{\partial}{\partial \boldsymbol{p}}\left[n\left(\boldsymbol{p}+\frac{\boldsymbol{s}}{2},\sigma\right)+n\left(\boldsymbol{p}-\frac{\boldsymbol{s}}{2},\sigma\right)\right] +\frac{1}{2}\frac{\partial}{\partial \boldsymbol{s}}\left[n\left(\boldsymbol{p}+\frac{\boldsymbol{s}}{2},\sigma\right)-n\left(\boldsymbol{p}-\frac{\boldsymbol{s}}{2},\sigma\right)\right], \tag{A6}$$

$$\frac{d\boldsymbol{s}}{d\sigma}=\boldsymbol{\eta}. \tag{A7}$$

$$\frac{d\boldsymbol{\eta}}{d\sigma}=\frac{1}{2}\frac{\partial}{\partial \boldsymbol{p}}\left[n\left(\boldsymbol{p}+\frac{\boldsymbol{s}}{2},\sigma\right)-n\left(\boldsymbol{p}-\frac{\boldsymbol{s}}{2},\sigma\right)\right] +\frac{\partial}{\partial \boldsymbol{s}}\left[n\left(\boldsymbol{p}+\frac{\boldsymbol{s}}{2},\sigma\right)+n\left(\boldsymbol{p}-\frac{\boldsymbol{s}}{2},\sigma\right)\right]. \tag{A8}$$

Next, according to the spirit of the expansions performed in the main text, we replace $\boldsymbol{s}$ by $\boldsymbol{s}/k$. and $\boldsymbol{\eta}$ by $\boldsymbol{\eta}/k$, and expand the functions in (A5)–(A8) into the inverse powers of k. Retaining only the main order terms, we arrive at the following set of equations:

$$\frac{d\boldsymbol{p}}{d\sigma}=\boldsymbol{\rho}, \tag{A9}$$

$$\frac{d\boldsymbol{\rho}}{d\sigma}=\boldsymbol{\nabla}_{\boldsymbol{\rho}} n(\boldsymbol{p},\sigma), \tag{A10}$$

$$\frac{d\boldsymbol{s}}{d\sigma}=\boldsymbol{\eta}, \tag{A11}$$

$$\frac{d\boldsymbol{\eta}}{d\sigma}=\tfrac{1}{2}\{\boldsymbol{\nabla}_{\boldsymbol{p}}\cdot[\boldsymbol{\nabla}_{\boldsymbol{p}} n(\boldsymbol{p},\sigma)\cdot\boldsymbol{s}]+[\nabla^2_{\boldsymbol{p}} n(\boldsymbol{p},\sigma)\boldsymbol{s}]\}, \tag{A12}$$

which represent Eqs. (31a)–(31d) of the main text.

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